Research Cycle 08: General Linear Model

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What is the "General Linear Model" (GLM)?

Definition (General Linear Model or GLM)

A general mathematical framework for expressing relationships among variables

- Differs from the "cookbook" approach to statistics
  - *t*-test, ANOVA, ANCOVA, χ^2 test, regression, correlation, etc.
- Can express/test linear relationships between a numerical dependent variable and any combination of independent variables (categorical or continuous)
- Can even be generalized to categorial dependent variables (through "Generalized Linear Models"; NB: advanced)

ANOVA, Regression, ANCOVA

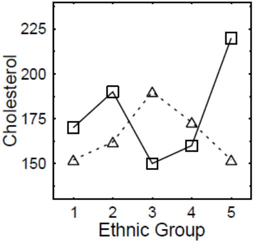


Fig 1a. Cholesterol levels by ethnic group and gender (male=sqr, female=tri).

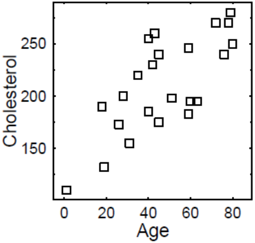


Fig 1a. Cholesterol levels by age.

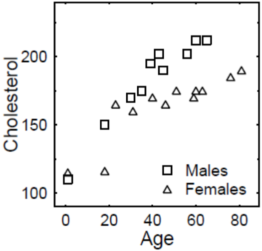


Fig 1a. Cholesterol levels by age and gender.

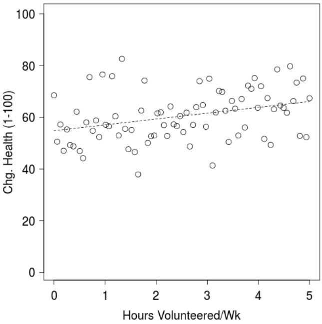
How the GLM represents relationships

Component of GLM	Notation
DV	Y
Grand Average	μ "mu"
Main Effects	$A, B, C, \ldots$
Interactions	$AB, AC, BC, ABC, \ldots$
Random Error	$S(Group)$

$$\begin{array}{ccccccccc} \text{Score} & = & \text{Grand Avg.} & + & \text{Main Effects} & + & \text{Interactions} & + & \text{Error} \\ Y & = & \mu & + & A + B + C + \ldots & + & AB + AC + BC + ABC + \ldots & + & S(Group) \end{array}$$

- Components of the model are estimated from the observed data
- Tests are performed (F) to see whether its variability is too large to be introduced by chance

An example: Simple Linear Regression



$$Y_i = \mu + b \times X_i + e_i$$
$$\text{Score}_i = \text{Baseline} + \text{Slope} \times \text{Hours}_i + \text{Error}_i$$
$$Y_i = 50 + 3 \times X_i + e_i$$
$$e_i \sim N(\mu = 0, \sigma^2 = 10)$$

Making comparisons across groups

Example (Spelling)

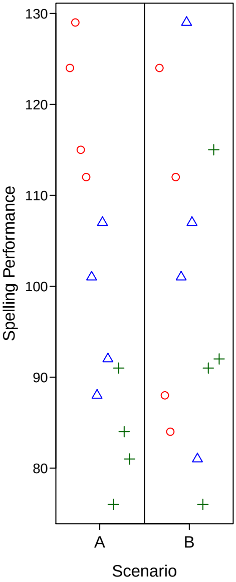
You wish to compare the benefits of three different spelling programs. Do these programs yield differences in spelling performance?

$H_0 : \mu_1 = \mu_2 = \mu_3$

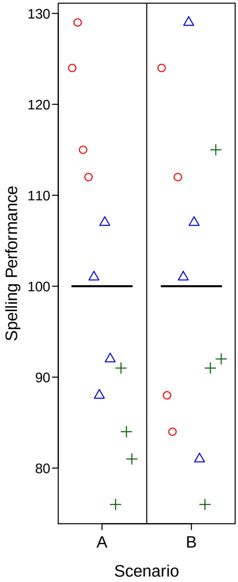
Factors and Levels

Factor: a categorical variable that is used to divide subjects into groups, usually to draw some comparison. Factors are composed of different *levels*. Do not confuse factors with levels!

Means, Variability, and Deviation Scores

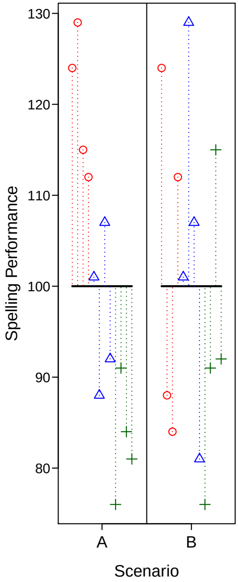


Means, Variability, and Deviation Scores



$$Y_{..} = \frac{\sum_{ij} Y_{ij}}{N}$$

Means, Variability, and Deviation Scores

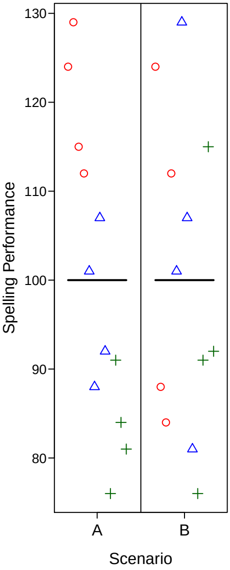


grand mean $Y_{..} = \frac{\sum_{ij} Y_{ij}}{N}$

$$SD_Y = \sqrt{\frac{\sum_{ij}\left(Y_{ij} - Y_{..}\right)^2}{N}}$$

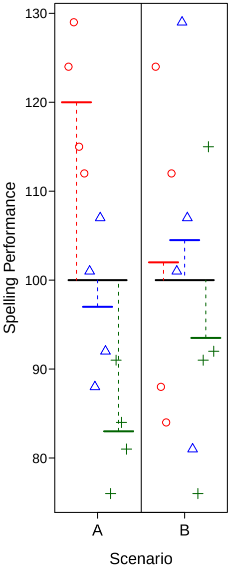
deviation score: $Y_{ij} - Y_{..}$

GLM for One-Factor ANOVA



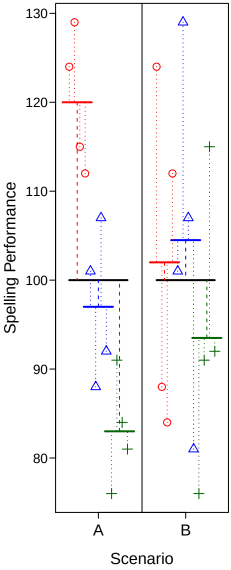
$$Y_{ij} = \mu$$

GLM for One-Factor ANOVA



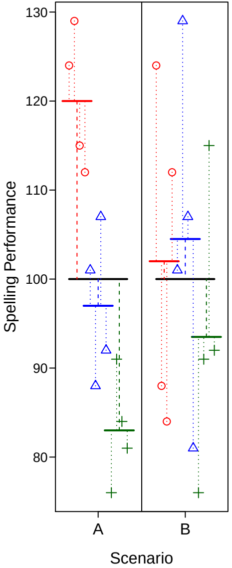
$$Y_{ij} = \mu + A_i$$

GLM for One-Factor ANOVA



$$Y_{ij} = \mu + A_i + S(A)_{ij}$$

GLM for One-Factor ANOVA



$$Y_{ij} = \mu + A_i + S(A)_{ij}$$

Estimation Equations

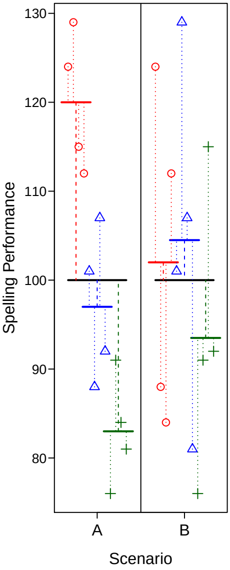
$$\hat{\mu} = Y_{..}$$

$$\hat{A}_i = Y_{i.} - \hat{\mu}$$

$$\widehat{S(A)}_{ij} = Y_{ij} - \hat{\mu} - \hat{A}_i$$

Note that $\sum_i \hat{A}_i = 0$ and $\sum_{ij} \widehat{S(A)}_{ij} = 0$

Sources of Variance



$$Y_{ij} = \mu + A_i + S(A)_{ij}$$

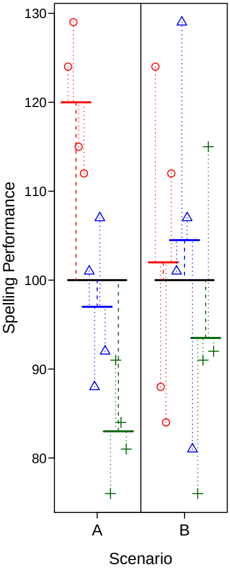
$$\begin{aligned} Y_{ij} - \mu &= A_i + S(A)_{ij} \\ \textit{individual} &= \textit{group} + \textit{random} \end{aligned}$$

Sum of Squares (SS)

A measure of variability consisting of the sum of squared *deviation* scores, where a deviation score is a score minus a mean.

$$SS_A = \sum (Y_{i.} - \mu)^2$$

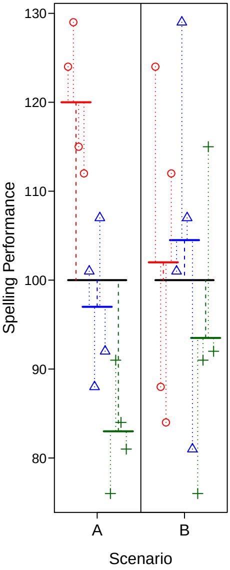
Decomposition Matrix



$$\hat{\mu} = 100$$
$$\hat{A}_1 = 120 - 100 = 20$$
$$\hat{A}_2 = 97 - 100 = -3$$
$$\hat{A}_3 = 83 - 100 = -17$$

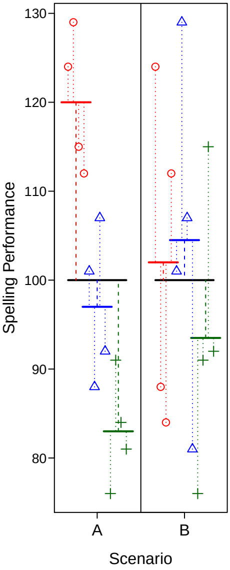
	Y_{ij}	=	$\hat{\mu}$	+	$\hat{A}_i$	+	$\widehat{S(A)}_{ij}$
	124	=	100	+	20	+	4
	129	=	100	+	20	+	9
	115	=	100	+	20	+	-5
	112	=	100	+	20	+	-8
	101	=	100	+	-3	+	4
	88	=	100	+	-3	+	-9
	107	=	100	+	-3	+	10
	92	=	100	+	-3	+	-5
	76	=	100	+	-17	+	-7
	91	=	100	+	-17	+	8
	84	=	100	+	-17	+	1
	81	=	100	+	-17	+	-2
$SS =$	123318	=	120000	+	2792	+	526

Logic of ANOVA



- Compare two estimates of the variability, the *between-group* estimate (SS_{between}) and the *within-group* estimate (SS_{within})
- If $H_0 : \mu_1 = \mu_2 = \mu_3$ is true, then these two measures estimate the same quantity.
- The extent to which the between-group variability exceeds the within-group variability gives evidence against $H_0 : \mu_1 = \mu_2 = \mu_3$.

Calculating SS_{between} and SS_{within}



Y_{ij}	=	$\hat{\mu}$	+	$\hat{A}_i$	+	$\widehat{S(A)}_{ij}$
124	=	100	+	20	+	4
129	=	100	+	20	+	9
115	=	100	+	20	+	-5
112	=	100	+	20	+	-8
101	=	100	+	-3	+	4
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107	=	100	+	-3	+	10
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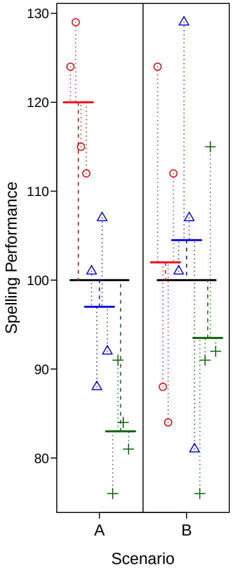
91	=	100	+	-17	+	8
84	=	100	+	-17	+	1
81	=	100	+	-17	+	-2
$SS =$ 123318	=	120000	+	2792	+	526

check your math

$$SS_Y = SS_\mu + SS_A + SS_{S(A)}$$

H_0 and Sums of Squares



$$Y_{ij} - \mu = A_i + S(A)_{ij}$$

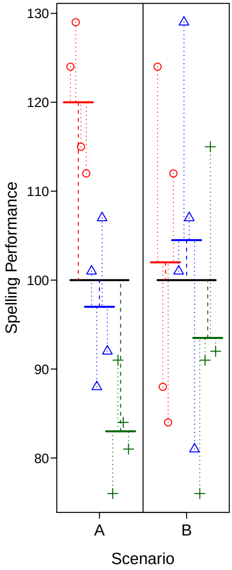
Scenario A

$SS_A = 2792$
$SS_{S(A)} = 526$
$SS_A + SS_{S(A)} = 3318$

Scenario B

$SS_A = 266$
$SS_{S(A)} = 3052$
$SS_A + SS_{S(A)} = 3318$

Mean Square and Degrees of Freedom



Degrees of Freedom (df)

The number of observations that are "free to vary".

$df_A = K - 1$

$df_{S(A)} = N - K$

where N is the number of subjects and K is the number of groups.

Mean Square (MS)

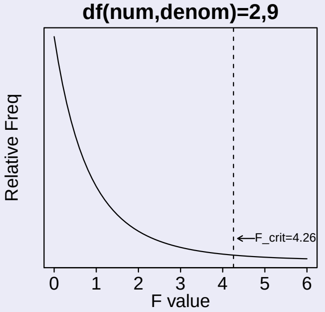
A sum of squares divided by its degrees of freedom.

$MS_A = \frac{SS_A}{df_A} = \frac{2792}{2} = 1396$

$MS_{S(A)} = \frac{SS_{S(A)}}{df_{S(A)}} = \frac{526}{9} = 58.4$

The F-ratio

F density function



If $F_{obs} > F_{crit}$, then reject H_0

F ratio

A ratio of mean squares, with df_{numerator} and df_{denominator} degrees of freedom.

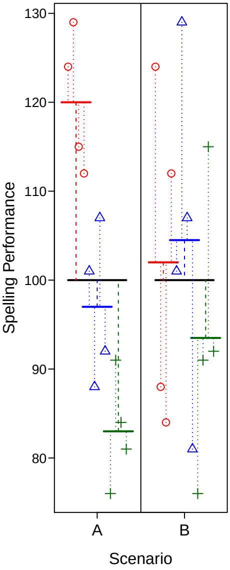
$$F_A = \frac{MS_A}{MS_{S(A)}} = \frac{1396}{58.4} = 23.886$$

df in denominator	df in numerator							
	1	2	3	4	5	6	7	8
1	161.40	199.50	215.70	224.60	230.20	234.00	236.80	238.90
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23

Density/Quantile functions for F-distribution

name	function
`pf(x, df1, df2, lower.tail = FALSE)`	density (get p given F_{obs})
`qf(p, df1, df2, lower.tail = FALSE)`	quantile (get F_{crit} given p)

Summary Table



Scenario A

Source	df	SS	MS	F	p	Error
μ	1	120000	120000.0	2053.232	<.001	$S(A)$
A	2	2792	1396.0	23.886	<.001	$S(A)$
$S(A)$	9	526	58.4			
Total	12	123318				

Scenario B

Source	df	SS	MS	F	p	Error
μ	1	120000	120000.0	353.878	<.001	$S(A)$
A	2	266	133.0	.392	.687	$S(A)$
$S(A)$	9	3052	339.1			
Total	12	123318				

Overview of One-Way ANOVA

1. Write the GLM: $Y_{ij} = \mu + A_i + S(A)_{ij}$
2. Write down the estimating equations:
   - $\hat{\mu} = Y_{..}$
   - $\hat{A}_i = Y_{i.} - \hat{\mu}$
   - $\widehat{S(A)}_{ij} = Y_{ij} - \hat{\mu} - \hat{A}_i$
3. Compute estimates for all terms in model.
4. Create *decomposition matrix.*
5. Compute *SS*, *MS*, *df*.
   - $df_\mu = 1$
   - $df_A = K - 1$
   - $df_{S(A)} = N - K$
   - $MS = SS/df$
6. Construct a summary ANOVA table.
7. Compare F_{obs} with F_{crit}.

R

use the `aov()` function, e.g.:

```
spelling$A <- factor(spelling$A)
mod <- aov(Y ~ A, data = spelling)
summary(mod)
```

http://talklab.psy.gla.ac.uk/stats/onefactoranova.html#sec-3-2

Other GLMs

- one-sample *t*-test $Y_i - c = \beta_0 + e_i$
- two-sample *t*-test $Y_i = \beta_0 + \beta_1 X_i + e_i$
  - where $X_i \in (0, 1)$
- paired-samples t-test $Y_{1i} - Y_{2i} = \mu + e_i$
- simple linear regression $Y_i = \beta_0 + \beta_1 X_i + e_i$
- multiple regression $Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + e_i$
- ANCOVA $Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 X_{1i} X_{2i} + e_i$
  - where $X_{1i} \in (0, 1)$ and $X_{2i} \in \mathbb{R}$